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BANACH CONTRACTION MAPPING PRINCIPLE FIXED POINT THEOREM AND its APPLICATION

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Banach's Fixed Point Theorem, also known as the Contraction Theorem, concerns certain mappings (so-called contractions) of a complete metric space into itself. It states conditions sufficient for the existence and uniqueness of a fixed point, which we will see is a point that is mapped to it. The theorem also gives an iterative process by which we can obtain approximations to the fixed point along with error bounds. The contraction mapping principle is one of the most useful tools in the study of nonlinear equations, be they algebraic equations, integral or differential equations. The principle is a fixed point theorem which guarantees that a contraction mapping of a complete metric space to itself has a unique fixed point which may be obtained as the limit of an iteration scheme defined by repeated images under the mapping of an arbitrary starting point in the space. As such, it is a constructive fixed point theorem and, hence, may be implemented for the numerical computation of the fixed point.

Keywords: *Banach's,, Contraction, Principle Fixed*

INTRODUCTION

One of the most useful and effective instruments in the usage of fixed point theorems is the study of metric space. The majority of nonlinear analysis and topology scholars are now exploring a broad variety of topics. Nowadays, a large variety of literature in this field is readily available. M. Frechet, a mathematician, was the first to establish the concept of metric space in 1906. The Hausdorff continues to refine this concept. Various authors continue their research in 1914, developing the metric space and certain fixed point theorems under various contraction conditions, based on the more practical formulation for fixed point theorems.

In the formulation of the fixed point theorems for different classes of contraction mappings, several topological spaces such as Metric space, full metric space, Banach space, Hilbert space, locally convex and Hausdorff space were used.

The researcher discovers the necessity to examine non-expansive mappings and stringent fixed point theorems in this area. Ciric's fixed point theorems, Khamsi's fixed point theorems, Brouwer fixed point theorem, Brouwer fixed point theorem, fixed point theorems for multi valued contractive mappings, contraction mapping principle, Edelstain fixed point theorem, Kranosekii fixed point theorem, Quasi non expansive mappings, Hahn Banach theorem Multivalued mappings for fixed points in Complete metric space, Bnyd and Wong's fixed point theorem' The researchers are also looking at certain typical fixed spots.

The preceding theorems and their applications are of interest to the researcher. In addition, the researcher will adapt the fixed point theorem in Metric and Banach spaces, as well as investigate various fixed point theorems for proving fixed points and the Banach contraction mapping principle and fixed point theorems with applications. In this study, the researcher will also look into fixed points, using a well-known classical finding concerning the existence and uniqueness of a fixed point of contrition, which is a function F from metric space X to itself that meets the requirement.

$$d(F(x),F(y)) \leq r\, d(x,y)$$

For all x,y in X.where r is such that, $0 \leq r \leq 1$

By using certain contractive conditions, we will apply this conclusion to the solution fixed point theorem.

OBJECTIVES

1. To show that in entire metric space, there are fixed points of maps that meet the contraction type requirement.
2. To show that in entire metric space, the uniqueness fixed point of maps meeting the contraction type requirement is unique.

EXAMPLE OF BANACH CONTRACTION PRINCIPLE

One of the most essential tools in the study of non-linear issues is the Banach contraction mapping theorem. It is a powerful example of the unifying power of functional analysis in an analytic approach, as well as the use of fixed point theorems in analysis. As a result, throughout the last four decades, several modifications of this theorem have been established by weakening its hypothesis while keeping the convergence feature of consecutive iterates to the unique fixed point of the mapping.

The concepts of no expansive and contractive mappings are crucial to these extensions. Another essential aspect of this principle's extension is the shared fixed point of a pair of mappings or a series of mappings that meet contractive type requirements.

One of the most intriguing extensions of the Banach contraction principle is to substitute a real valued function with values smaller than unity for the Lipschitz constant k. The following theorem by Rakotch [2013] was one of the first generalisations of Banach's contraction principle to gain widespread recognition.

Theorem: Let (X, d) be a complete metric space and suppose T: X X satisfies $d(Tx, Ty) \le \alpha(d(x, y)).d(x, y), \forall x, y \in X$, Where α: $[0,\infty) \to [0, \infty)$

Is monotonically decreasing Then, T has a unique fixed point z and for all $x_0 \in X$ we have, $T^n x_0 \to z$ as $n \to \infty$. Rokotch's theorem is related to the following theorem by Edelstein [1962].

Theorem: Let (X, d) be a non empty compact metric space and suppose T: X X satisfies $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$. Then T has a unique fixed point z and for all $x_0 \in X$ we have $T^n x_0 \to z$ as $n \to \infty$. Bailey [1966] extended the result of Edelstein [1962] to compact metric space in the following theorem.

Theorem: Let (X, d) be a compact metric space and T: X X be continuous. If there exists n = n (x, y) with d $(T^n x, T^n y) < d(x, y)$ for $x \ne y$, then has a unique fixed point. A subsequent generalization of Rakotch's result was obtained by Boyd and Wong [1969].

Theorem: Let (X, d) be a non empty complete metric space and suppose $T: X \to X$ satisfies $(Tx, Ty) \le \phi(d(x, y))$ for all $x, y \in X$ where ϕ: $[0, \infty) \to [0, \infty)$ is upper semi continuous from the right and satisfies $0 \le \phi(t) < t$ for $t > 0$. then T has a unique fixed point z and for all $x_0 \in X$ we have $T^n x_0 \to z$ as $n \to \infty$. . A quantitative variant of the Boyd – Wong [1969] theorem was proved by Browder [1968].

CONCEPT OF FIXED POINT:

Fixed point theory is a prominent issue in nonlinear functional analysis, and it has some interesting applications in other disciplines of mathematics for establishing existence and uniqueness theorems for nonlinear problems.

The fixed point theorems for several classes of contraction mapping have been formulated using various topological spaces such as metric, 2-metric, Banch, Hilbert, and Housdroff spaces, among others.

Definition A fixed point of a function "f" from a set S to itself is a point x in S such that $f(x) = x$.

Fixed points arise naturally while solving operator equations.

Example: Consider the integral equation

$$y(s) = y_0(s) + \int_a^b f(s, u, y(u))du, \qquad s \in [a, b]$$

Where y_0 is given continuous real valued function on [a,b]

Let S denote the set of all real valued continuous function on $[a, b]$ and

$$F(y)(s) = x_0(s) + \int_a^b f(s,u,y(u))du \quad for \ y \in S, s \in [a,b] \qquad (2.1)$$

If for each $s \in [a,b]$, the function $f(s,u,y(u))$ is an integral function of u on $[a,b]$. The function $\int_a^b f(s,u,y(u))du$ is a continuous function of S on $[a,b]$, then $F(y) \in S$. Clearly x is a solution of this integral equation if it is fixed point of the function F.

Now comes the study of fixed points, which begins with a well-known classical conclusion regarding the existence and uniqueness of a fixed point of a contraction that is a function F from metric space X.

to itself satisfying $\varrho(F(x),F(y)) \leq r\,\varrho(x,y)$ for all $x,y \in X$ and r with a value of 0 By changing it to an integral equation of type equation 2.1, we were able to apply this finding to the solution of an initial value issue. The basic goal in fixed point theory is to determine the points that are invariant under the action of the mapping in question, i.e. to find the functional solution.

Equation $f(x) = x$ in the appropriate function space.

Example 1.1.2 If we consider the function $f:[0,1] \to [0,1]$ defined by $f(x) = x^2, x \in [0,1]$.

Then in this case only fixed points of function f are 0 and 1 because $f(0) = 0$ and $f(1) = 1$ it is possible that a mapping may not have a single fixed point or may have a number of fixed points.

In the mapping $f: R^+ \to R^+$ defined by $f(x) = 2x+1, \ x \in R^+$ does not have any fixed point and identity mapping $I: R \to R$ has infinite numbers of fixed points.

CONCEPT OF MAPPINGS OR FUNCTIONS:

In mathematics, the notion of mapping one set into another is very important. The concept of mapping has been with us from the beginning of our higher mathematic education. The following definition helps clarify the notion of function or mapping.

Definition: Assume that A and B are two non-empty sets. If R ties every element in set A with a unique element in set B, then a relation A defined from a set to a set is termed a function.

A mapping or map is another name for a function. One of the most basic ideas in mathematics is the rotation of a mapping or function. The following is an example of an intuitive mapping concept: “A mapping is a relationship that connects each member of set A to a single element of another set B."

TYPES OF MAPPINGS:

We define various types of mappings as follows,

1. One-to-one mapping (injective mapping).
2. Maps that is subjective (onto function).
3. Bijective mapping (one – one as well as onto function).
4. Inverse mapping is the fourth option.
5. Constant mapping
6. Mapping that isn't continuous.
7. Mapping in linear form.

Nonlinear mapping is a kind of nonlinear mapping.

Definition: A function $f : A \to B$ is called one–one function if $x_1, x_2 \in A$, $x_2 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

As a result, a function is one–one if it connects various domain components with different co-domain elements. Injective mapping is another name for this one–one function.

Example: Because various nations have different capitals, a function that relates each country in the world, its capital, is one – one.

Definition: A function $f : A \to B$ is said to be an onto function or a surjective if its range is equal to it's co-domain. Such onto function is called surjective mapping.

Example: If A $\{1, 2, 3\}, B = \{2, 4, 6\}$ and $f : A \to B$ is defined by, $f : \{(1,6), (2,4), (3,2)\}$

Here range of function $f = \{6, 4, 2\} = B =$ codomain of f.

Function f is onto.

Definition: A function is said to be bijective if it is one – one and onto.

Example: If function $f: \mathrm{R} \to R$ defined by $f(\mathrm{x}) = \mathrm{x}$. Then f is one-one and onto.

Definition: Let A and B be two non–empty set and $f\ A \to B$ is a mapping. If we define a mapping from B to A by using function f is called Inverse function or inverse mapping and it is denoted by f^{-1}.

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If $f : A \xrightarrow[onto]{one-one} B$ $then\ f^{-1}: B \xrightarrow[onto]{one-one} A.$

Example: $f\{(1,1),(2,4),(3,9),(4,16)\}$ then,

$f^{-1} = \{(1,1),(4,2),(9,3),(16,4)\}$. Thus if $f(x) = y$ then $f^{-1}(y) = x.$

Definition: A function f is said to be continuous at $x = a$ if

1) $f(a)$ exist.
2) $\lim_{x\to a} f(x)\ exists\ and\ \lim_{x\to a} f(x) = f(a)$

Continuous mapping is the term for this kind of mapping. If X has a limited number of dimensions. Then there's an infinitely long nonlinear mapping from X to Y.

Definition: If a function f is not continuous at $x = a$ then such a function is called discontinuous function or discontinuous mapping at $x = a$.

If X is infinite dimensional, then the nonlinear mapping from X to Y is discontinuous.

Definition: A given mapping f is linear mapping from a set A to a set B if

$f(x + y) = f(x) + f(y)$ and $f(\propto x) = \propto f(x)\ for\ all\ x, y, \propto\ \epsilon\ A.$

Definition: When the mapping is not linear, called non linear mapping.

Example $f(x) = x^2,\ f(x) = \sqrt{x}, f(x) = \int_0^1 x^2(t)dt$, are the examples of nonlinear mappings.

CONTRACTION MAPPING THEOREM

Let f: X X denote a mapping from one set to another. If f(x) = x, we designate a point x X a fixed point of f. If [a, b] is a closed interval, then any continuous function f: [a, b] f: [a, b] f: [a, b] f: [a, b] f: [a, b] f: [a, b] f: [a, b] f: [a, b] f: [a, As follows, this is a result of the intermediate value theorem. Because f(a) a and f(b) b are equal, we obtain f(b) b 0 f(a) a. Because the difference f(x) x is continuous, the intermediate value theorem says that for any x [a, b], 0 is a value of f(x)x, and that x is a fixed point of f. There might, of course, be more than one fixed point.

The most fundamental fixed-point theorem in analysis will be discussed here. It was created by Banach and published in his Ph.D. thesis (1920, published in 1922).

Theorem Let *(X, d)* be a complete metric space and f : X → X be a map such that

$$d(f(x), f(x')) \leq cd(x, x')$$

for some 0 ≤ c < 1 and all x and x 0 in X. Then f has a unique fixed point in X. Moreover, for any $x_0 \in X$ the sequence of iterates $x_0, f(x_0), f(f(x_0)), \ldots$ converges to the fixed point of f.

When d $(f(x), f(x')) \leq cd(x, x')$ for some $0 \leq c < 1$ A contraction is defined as all x and x 0 in X. For all pairings of points, a contraction lowers distances by a consistent factor c smaller than 1. Theorem 1.1 is also known as Banach's fixed-point theorem or the contraction mapping theorem.

The Newton's approach is used to estimate a solution in R to the numerical equation g(x) = 0, where g is differentiable: determine an approximate solution x0 and then calculate the recursively specified sequence.

$$x_n = x_{n-1} - \frac{g(x_{n-1})}{g'(x_{n-1})}.$$

This recursion amounts to iteration of the function $f(x) = x - g(x)/g'(x)$ starting from $x = x_0$. A solution of $g(x) = 0$ is the same as a solution of $f(x) = x$: a fixed point of f. To use Newton's method to estimate $\sqrt{3}$, we take $g(x) = x^2 - 3$ and seek a (positive) root of g(x). The Newton recursion is

$$x_n = x_{n-1} - \frac{x_{n-1}^2 - 3}{2x_{n-1}} = \frac{1}{2}\left(x_{n-1} + \frac{3}{x_{n-1}}\right),$$

so $f(x) = (1/2)(x + 3/x)$. The fixed points of f are the square roots of 3. In the following table are iterations of f with three different positive choices of x0.

n	x_n	x_n	x_n
0	1.5	1.9	10
1	1.75	1.7394736842	5.15
2	1.7321428571	1.7320666454	2.8662621359
3	1.7320508100	1.7320508076	1.9564607317
4	1.7320508075	1.7320508075	1.7449209391
5	1.7320508075	1.7320508075	1.7320982711

All three iterate sequences in the table seem to have a tendency to 3 1.7320508075688. Because the last x0 in the table is a long distance from 3, it takes longer for the iterations to start approximating 3.

We need to discover a full metric space on which f(x) = (1/2)(x+3/x) is a contraction to justify applying the contraction mapping theorem to this situation. The set (0,) does not function since it is incomplete. For all t > 0, the closed interval Xt = [t,) is complete. Which t has the form f(Xt) Xt and f is a contraction on Xt? Well, the minimum of f(x) on (0,) is at x = 3, and f(3) = 3, so for every t 3 we get x t = f(x) 3 t, and so f(Xt) Xt. For each positive x and x 0, we have to identify a t for which f is a contraction on Xt.

$$f(x) - f(x') = \frac{x - x'}{2}\left(1 - \frac{3}{xx'}\right).$$

If x ≥ t and x 0 ≥ t then 1 − 3/t2 ≤ 1 − 3/xx0 < 1. Therefore |1 − 3/xx0 | < 1 as long as 1−3/t2 > −1, which is equivalent to t 2 > 3/2 Taking p 3/2 < t ≤ √ 3 (e.g., t = 3/2 = 1.5), we have f: Xt → Xt and |f(x) − f(x 0)| ≤ (1/2)|x − x 0 | for x and x 0 in Xt . The contraction mapping theorem says the iterates of f starting at any x0 ≥ t will converge to √ 3. How much iteration is needed to approximate √ 3 to a desired accuracy will be addressed in Section 2 after we prove the contraction mapping theorem. Although (0, ∞) is not complete, iterations of f starting from any x > 0 will converge to √ 3; if we start below √ 3 then applying f will take us above √ 3 (because f(x) ≥ √ 3 for all x > 0) and the contraction mapping theorem on [√ 3, ∞) then kicks in to guarantee that further iterations of f will converge to √ 3.

The contraction mapping theorem has a wide range of applications in analysis, both theoretical and practical. We will provide one key application after establishing the theorem (Section 2) and considering various extensions (Section 3). Picard's theorem (Section 4) is the fundamental existence and uniqueness theorem for ordinary differential equations. The contraction mapping theorem has been used to partial differential equations, the Gauss–Seidel method of solving systems of linear equations in numerical analysis, a demonstration of the inverse function, and Google's Page Rank algorithm, among other things. A rudimentary introduction to the concepts of iteration and contraction in analysis, together with a few examples, is now available, as well as a thorough treatment.

CONCLUSION

Most of the fixed point theorems included in the thesis are constructive theorems. They contain iterations leading to the resulting fixed points. Have close structural similarities. The iterations leading to the fixed points are also the same. But the theorems in complete Menger Spaces with continuous t-norm and on d-complete topological spaces cannot be deduced from the similar theorems proved in F-complete topological spaces. This is due to the fact that although F-complete topological spaces are generalisations of complete Menger Spaces with continuous t-norm and also of d-complete topological spaces, there are. Some properties which may hold in the former space but are never satisfied in the context of the latter two spaces. The differences in the mathematical structures of the above spaces are thus revealed by these theorems. A random iteration scheme has been presented in the thesis. We have used this random iteration in finding random fixed points of certain random operators. This iteration scheme may be applied to?! Different kinds of operators and] its convergences to random fixed points may further be studied. Z1 A necessary and sufficient condition for the existence of unique fixed points for a clasjs of self-mappings on F-complefce topological spaces, has been obtained. We have proved the corresponding results on complete Menger Spaces with t 3 min, on d-complete topological spaces and on complete metric spaces. By the application of the sufficiency condition we have obtained certain fixed point theorems on metric space as our corollaries.

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